ON THE CHAOS GAME OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. Every attractor of an iterated function system (IFS) of continuous functions on a first-countable Hausdorff topological space satisfies the probabilistic chaos game. By contrast, we prove that the backward minimality is a necessary condition to get the deterministic chaos game. We obtain that an IFS of homeomorphisms of the circle satisfies the deterministic chaos game if and only if it is forward and backward minimal. This provides examples of attractors that do not satisfy the deterministic chaos game. We also prove that every contractible attractor (in particular strong-fibred attractors) satisfies the deterministic chaos game.

1. Introduction

Within fractal geometry, iterated function systems (IFSs) provide a method for both generating and characterizing fractal images. An iterated function system (IFS) can also be thought of as a finite collection of functions which can be applied successively in any order. Attractors of this kind of systems are self-similar compact sets which draw any iteration of any point in an open neighborhood of itself. There are two methods of generating the attractor: deterministic, in which all the transformations are applied simultaneously, and random, in which the transformations are applied one at a time in random order following a probability. The chaos game, popularized by Barnsley [3], is the simple algorithm implementing the random method. We have two different forms to run the chaos game. One involves taking a starting point and then choose randomly the transformation on each iteration accordingly to the assigned probabilities. The other one starts by choosing a random order iteration and then applying this orbital branch anywhere in the basin of attraction. The first form of implementation is called *probabilistic chaos game* [7, 6]. The second implementation is called deterministic chaos game (also called disjunctive chaos game) [4, 10]. In this paper we show that every IFS of continuous maps on a first-countable Hausdorff topological space satisfies the probabilistic chaos game (see also [6]) and give necessary and sufficient conditions to get the deterministic chaos game. As an application we obtain that an IFS of homeomorphisms of the circle satisfies the deterministic chaos game if and only if it is forward and backward minimal which provides examples of attractors that do not satisfy the deterministic chaos game. We also prove that every contractible attractor (in particular strong-fibred attractors) satisfies the deterministic chaos game.

Key words and phrases. iterated function system, contractible attractor, deterministic and probabilistic chaos game, forward and backward minimality.

1.1. **Iterated function systems.** Let X be a Hausdorff topological space. We consider a finite set $\mathscr{F} = \{f_1, \dots f_k\}$ of continuous functions from X to itself. Associated with this set \mathscr{F} we define the *semigroup* $\Gamma = \Gamma_{\mathscr{F}}$ generated by these functions, the *Hutchinson operator* $F = F_{\mathscr{F}}$ on the hyperspace $\mathscr{H}(X)$ of the non-empty compact subsets of X endowed with the Vietoris topology

$$F: \mathcal{H}(X) \to \mathcal{H}(X), \qquad F(A) = \bigcup_{i=1}^k f_i(A)$$

and the *one-step skew-product* $\Phi = \Phi_{\mathscr{F}}$ on the product space of $\Omega = \{1, \dots, k\}^{\mathbb{N}}$ and X

$$\Phi: \Omega \times X \to \Omega \times X$$
, $\Phi(\omega, x) = (\sigma(\omega), f_{\omega_1}(x))$

where $\omega = \omega_1 \omega_2 \cdots \in \Omega$ and $\sigma : \Omega \to \Omega$ is the lateral shift map. The action of the semigroup Γ on X is called *iterated function system* generated by f_1, \ldots, f_k (or, by the family \mathscr{F} for short). Finally, given $\omega = \omega_1 \omega_2 \cdots \in \Omega$ and $x \in X$,

$$O_{\omega}^+(x) = \{f_{\omega}^n(x) : n \in \mathbb{N}\}$$
 where $f_{\omega}^{n \text{ def}} f_{\omega_n} \circ \cdots \circ f_{\omega_1}$ for every $n \in \mathbb{N}$,

are, respectively, the ω -fiberwise orbit of x and the orbital branch corresponding to ω . We introduce now some different notions of invariant and minimal sets and after that we give the definition of attractor. In what follows A denotes any subset of X.

1.2. **Invariant and minimal sets.** We say that A is forward or backward *invariant* set for the IFS if $f(A) \subset A$ or $\emptyset \neq f^{-1}(A) \subset A$ for all $f \in \Gamma$ respectively. Another different notion of invariance can be introduced attending to the dynamics of the Hutchinson operator. Namely, we say that A is a *self-similar* set for the IFS (or F-invariant) if $A = f_1(A) \cup \cdots \cup f_k(A)$.

We say that the IFS is forward/backward *minimal* if the unique forward/backward invariant non-empty closed set is the whole space X. It is not difficult to see that forward minimality is equivalent to the density of any Γ -orbit. Recall that the (forward) Γ -orbit of a point $x \in X$ is the set $\Gamma(x) = \{g(x) : g \in \Gamma\}$. By extension, we will say that A is a forward *minimal set* if the closure of $\Gamma(x)$ contains A for all $x \in A$.

1.3. **Attractors.** We will introduce the notion of attractor following [7, 8, 5, 6].

The *pointwise basin* of a compact set A of X for F is the set $\mathcal{B}_p(A)$ of points $x \in X$ such that $F^n(\{x\}) \to A$ as $n \to \infty$. The convergence here is with respect to Vietoris topology, or equivalently, in the metric space case, with respect to Hausdorff metric [18, pp. 66–69].

Definition 1.1. A compact set A is said to be pointwise attractor for the Hutchinson operator F if there exists an open set U of X such that $A \subset U \subset \mathcal{B}_p(A)$.

A slightly strong notion of attractor is the following:

Definition 1.2. A compact set A is said to be strict attractor for the Hutchinson operator F if there exists an open neighborhood U of A such that

$$\lim_{n\to\infty} F^n(K) = A \text{ in the Vietoris topology for all compact set } K \subset U.$$
 (1)

The basin $\mathcal{B}(A)$ of an attractor A is the union of all open neighborhoods U such that (1) holds.

Examples of pointwise attractors that are not strict attractors can be find in [6]. By abuse of the terminology, it is customary to say pointwise/strict attractor for the IFS generated by \mathscr{F} rather than for the associated Hutchinson operator F.

We remark that it is usually to include in the definition of attractor that F(A) = A (cf. [8, Def. 2.2]). Under ours mild assumptions on X, it is unknown the continuity of the Hutchinson operator and thus, it is not, a priori, clear that A is a self-similar (F-invariant) set. Nevertheless, the following result proves that any attractor must be a forward minimal self-similar set and, in the case of a strict attractor, must attract any compact set in the basin of attraction which, a priori, is also not clear from the definition.

Theorem A. Consider the IFS generated by \mathcal{F} and let A be a compact set in X.

- (1) A is a forward minimal self-similar set for the IFS if and only if $A \subset \mathcal{B}_p(A)$. In particular, every pointwise attractor is a forward minimal self-similar set for the IFS.
- (2) If A is a strict attractor for the IFS then it is a pointwise attractor, $\mathcal{B}(A) = \mathcal{B}_p(A)$ and $\lim_{K \to \infty} F^n(K) = A$ in the Vietoris topology for all compact set $K \subset \mathcal{B}(A)$.
- 1.4. **Chaos game.** Now, we focus our study to the chaos game of pointwise attractors. First, we will give a rigourously definition of chaos game.

Following [7], we consider any probability \mathbb{P} on Ω with the following property: there exists $0 so that <math>\omega_n$ is selected randomly from $\{1, ..., k\}$ in such a way that the probability of $\omega_n = i$ is greater than or equal to p, regardless of the preceding outcomes, for all $i \in \{1, ..., k\}$ and $n \in \mathbb{N}$. More formally, in terms of conditional probability,

$$\mathbb{P}(\omega_n = i \mid \omega_{n-1}, \dots, \omega_1) \ge p.$$

Observe that Bernoulli measures on Ω are typical examples of these kinds of probabilities.

Definition 1.3. Let A be a pointwise attractor of the IFS generated by \mathscr{F} . We say that A satisfies the

(1) probabilistic chaos game if for any $x \in \mathcal{B}_p(A)$ there is $\Omega(x)$ with $\mathbb{P}(\Omega(x)) = 1$ such that

$$A \subset \overline{O_{\omega}^+(x)}$$
 for all $\omega \in \Omega(x)$.

(2) deterministic chaos game if there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$A \subset \overline{O_{\omega}^+(x)}$$
 for all $\omega \in \Omega_0$ and $x \in \mathcal{B}_p(A)$.

If the IFS is forward minimal we say that the IFS satisfies the probabilistic/deterministic chaos game.

The sequences in Ω with dense orbit under the shift map are called *disjunctives*. Notice that the set of such sequences have \mathbb{P} -probability one and its complement is a σ -porous set with respect to the Baire metric in Ω [4]. The following result shows that, in fact, the simple existence of a sequence ω such that every point in the basin of attraction has dense ω -fiberwise orbit in the attractor is enough to guarantee that for any disjunctive sequence we also draw the attractor. This brings to light that actually the deterministic chaos game does

not depend on the probability \mathbb{P} and explains the name *deterministic* (or disjuntive chaos game) since the disjuntive sequences are a priori well known.

Theorem B. Consider the IFS generated by \mathscr{F} and let A be a compact set in X and $x \in \mathcal{B}_{v}(A)$. Then,

(1) A and $\overline{\Gamma(x)}$ are forward invariant compact sets of X and $A \subset \overline{\Gamma(x)}$. In particular,

$$\overline{\{f_{\omega}^m(x): m \geq n\}}$$
 is a compact set for all $n \in \mathbb{N}$ and $\omega \in \Omega$;

(2) $A \subset \overline{O^+_{\omega}(x)}$ if and only if

$$\lim_{n\to\infty} \overline{\{f_{\omega}^m(x): m\geq n\}} = A \text{ in the Vietoris topology;}$$

(3) if A is a pointwise attractor, A satisfies the deterministic chaos game if and only if

there exists
$$\omega \in \Omega$$
 such that $A \subset \overline{O_{\omega}^+(x)}$ for all $x \in \mathcal{B}_p(A)$

and if and only if

$$A \subset \overline{O_{\omega}^+(x)}$$
 for all $x \in \mathcal{B}_p(A)$ and disjuntive sequence $\omega \in \Omega$.

1.4.1. *Probabilistic chaos game*. The relationship between the algorithm and the attractor is not at all obvious, as there is no evident connection between them. Initially, the method was developed for contracting IFSs [3]. Later, it was generalized to IFSs of continuous functions on proper metric spaces [7]. Recently in [6], Barnsley, Leśniak and Rypka proved the probabilistic chaos game for continuous IFSs on first-countable normal Hausdorff topological spaces (in fact they only need to assume that the attractor is first-countable). As a consequence of Theorem B, the assumption that the space is normal can be removed. Indeed, it suffices to note that for every $x \in \mathcal{B}_p(A)$ by Item (1) of Theorem B, the closure of the orbit of x is a forward invariant compact set and thus a normal Hausdorff topological space. Restricting the IFSs to this space, we are in the assumptions of [6] and hence one get the following:

Corollary C. Let A be a compact first-countable forward minimal self-similar set of the IFS generated by \mathscr{F} . Then for every $x \in \mathcal{B}_p(A)$ there exists $\Omega(x) \subset \Omega$ with $\mathbb{P}(\Omega(x)) = 1$ such that

$$A \subset \overline{O^+_{\omega}(x)}$$
 for all $\omega \in \Omega(x)$.

In particular, any pointwise attractor of an IFS of continuous maps of a first-countable Hausdorff topological space satisfies the probabilistic chaos game.

1.4.2. Deterministic chaos game. In the case of contractive IFSs a very simple justification of the deterministic chaos game can be given along the lines in [12, proof of Theorem 5.1.3]. In [9] it is also proved for weakly hyperbolic IFSs which are an extension of the previous contractive IFSs. Later, in [4] the deterministic chaos game was obtained for a more general class of attractors, the so-called strongly-fibred. An attractor A of an IFS on X is said to be strongly-fibred if for every open set $U \subset X$ such that $U \cap A \neq \emptyset$, there exists $\omega \in \Omega$ so that

$$A_{\omega} \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A) \subset U.$$

New examples of forward minimal IFSs satisfying the deterministic chaos game which are not strongly-fibred was given in [10]. The next result goes in the direction to provides necessarily conditions to yield the deterministic chaos game.

Theorem D. Any forward minimal IFS generated by continuous maps of a compact Hausdorff topological space that satisfies the deterministic chaos game must be also backward minimal.

As an application of the above result we can complete the main result in [10] obtaining the following corollary:

Corollary E. Let f_1, \ldots, f_k be circle homeomorphisms. Then the following statements are equivalent:

- (1) the IFS generate by f_1, \ldots, f_k satisfies the deterministic chaos game,
- (2) there exists $\omega \in \Omega$ such that $\overline{O_{\omega}^+(x)} = S^1$ for all $x \in S^1$,
- (3) the IFS generated by f_1, \ldots, f_k is forward and backward minimal.

Consequently this allows us to construct an contra-example of the deterministic chaos game for general IFSs. More specifically, any forward minimal but not backward minimal IFS of homeomorphisms of the circle does not satisfy the deterministic chaos game. Observe that for ordinary dynamical systems on the circle, the minimality of a map T is equivalent to that of T^{-1} . However this is not the case of IFS with more than one generator:

Corollary F. There exists an IFS of homeomorphisms of the circle that is forward minimal but not backward minimal. Consequently, there exists a strict attractor A of an IFS on a compact metric space such that A does not satisfies the deterministic chaos game.

The last main result of this paper is a generalization of [4]. In [4] it is proved that every strongly-fibred strict attractor of a complete metric space satisfies the deterministic chaos game. We are going to introduced a similar category of attractors:

Definition 1.4. We say that an attractor A of the IFS is contractible if for every compact set K in A so that $K \neq A$ and for every open cover \mathcal{U} of A, there exist $g \in \Gamma$ and $U \in \mathcal{U}$ such that $g(K) \subset U$.

It is not difficult to see that, in the metric space case, an attractor A is contractible if and only if for every compact set K in A so that $K \neq A$, there exists a sequence $(g_n)_n \subset \Gamma$ such that the diameter diam $g_n(K)$ converges to zero as $n \to \infty$. This equivalence motives the name of "contractible" since we can contract the diameter of any non-trivial closed set in A. Similarly, it is easy to show that strongly-fibred implies contractible (see Lemma 3.7). In fact, we will prove that if $f_i(A)$ is not equal to A for some generator f_i then both notions, strongly-fibred and contractible are equivalent. After this observation, we can say that the following theorem generalizes the main result in [4].

Theorem G. Any pointwise contractible attractor A of the IFS generated by \mathscr{F} satisfies the deterministic chaos game. Moreover, if either, A is either, strongly-fibred or the generators restricted to the attractor are homeomorphisms, then

 $\Omega \times A = \overline{\{\Phi^n(\omega, x) : n \in \mathbb{N}\}}$ for all disjunctive sequence ω and $x \in A$.

As a consequence of the above result we will prove the following:

Corollary H. Every forward and backward minimal IFS of homeomorphisms of a metric space so that the associated semigroup has a map with exactly two fixed points, one attracting and one repelling satisfies the deterministic chaos game.

Organization of the paper: In Section 2 we study the basin of attraction of pointwise/strict attractors and we prove Theorem A and the two first items of Theorem B. We complete the proof of this theorem in Subsection 3.1 where we study the deterministic chaos game. In Subsection 3.2 we prove Theorem D and in Subsection 3.3 we study the deterministic chaos game on the circle (Corollaries E and F).

Standing notation: In the sequel, X denotes a Hausdorff topological space. We assume that we work with an IFS of continuous maps f_1, \ldots, f_k on X and we hold the above notations introduced in this section.

2. On the basin of attraction

We start giving a basic topological lemma:

Lemma 2.1. *Let A and B be two compact sets in X.*

- (1) if $A \cap B = \emptyset$ then there exist disjoint open neighborhoods of A and B;
- (2) if $\{U_1, \ldots, U_s\}$ is a finite open cover of A then there exist compact sets A_1, \ldots, A_s in X so that

$$A = A_1 \cup \cdots \cup A_s$$
 and $A_i \subset U_i$ for $i = 1, \ldots, s$.

Proof. The first item is a well known equivalent definition of Hausdorff topological space (see [17, Lemma 26.4 and Exercice 26.5]). Hence, we only need to prove the second item. First of all, notice that it suffices to prove the result for an open cover of *A* with two sets. So, let {*U*₁, *U*₂} be an open cover of *A*. Since *X* is Hausdorff, *A* is a closed subset of *X*. Let us consider compact subsets $K_1 = A \setminus U_2 \subset U_1$ and $K_2 = A \setminus U_1 \subset U_2$. If $A \subset K_1 \cup K_2$ then we set $A_1 = K_1$ and $A_2 = K_2$ and we are done. Otherwise, $A \setminus (K_1 \cup K_2)$ is a nonempty subset of *A* and it is easy to see that $K_1 \cap K_2 = \emptyset$. Since *X* is Hausdorff and K_1 and K_2 are compact disjoint subsets of *X*, by the first item, there exist disjoint open subsets V_i of *X* so that $K_i \subset V_i$, for i = 1, 2. We may assume that $V_i \subset U_i$. Now let us take $A_1 = A \setminus V_2 \subset U_1$ and $A_2 = A \setminus V_1 \subset U_2$. Then A_1 and A_2 are compact subsets of *X* and $A = A_1 \cup A_2$ that concludes the proof. □

Let A, A_n for $n \ge 1$ be compact subsets of X. We recall that the Vietoris topology in $\mathcal{H}(X)$ is generated by the basic sets of the form

$$O(U_1,\ldots,U_m) = \{K \in \mathcal{H}(X) : K \subset U_1 \cup \cdots \cup U_m, K \cap U_i \neq \emptyset \text{ for } k = 1,\ldots,m\}$$

where $U_1, ..., U_m$ are open sets in X and $m \in \mathbb{N}$. Hence, if $A_n \to A$ in the Vietoris topology then $A_n \in O(U)$ for any n large enough and any open set U in X such that $A \subset U$. In particular $A_n \subset U$ for all n sufficiently large. Moreover, we get the following:

Lemma 2.2. $A_n \to A$ in Vietoris topology if and only if for any pair of open sets U and V such that $A \subset U$ and $A \cap V \neq \emptyset$, there is $n_0 \in \mathbb{N}$ so that

$$\bigcup_{n\geq n_0}A_n\subset U\quad and\quad V\cap A_n\neq\emptyset \ \ for\ all\ n\geq n_0.$$

In particular,

$$A = \bigcap_{m > 1} \overline{\bigcup_{n \ge m}} A_n. \tag{2}$$

Proof. Assume that $A_n \to A$ in the Vietoris topology. Let U be any open set such that $A \subset U$. Applying the above observation there exists $n_0 \in \mathbb{N}$ such that $A_n \subset U$ for all $n \ge n_0$. Now we will see that for any open set V with $A \cap V \ne \emptyset$, it holds that $A_n \cap V \ne \emptyset$ for all n sufficiently large. By the compactness of A, we extract open sets U_1, \ldots, U_s in X such that $A \cap U_i \ne \emptyset$ and $A \subset V \cup U_1 \cup \cdots \cup U_s$. Hence $O(V, U_1, \ldots, U_s)$ is an open neighborhood of A in $\mathscr{H}(X)$. Since A_n converges to A then $A_n \in O(V, U_1, \ldots, U_s)$ for all n large enough and in particular $A_n \cap V \ne \emptyset$ for all n large. We will prove the converse. Let $O(U_1, \ldots, U_m)$ be an basic open neighborhood of A. Thus, U_1, \ldots, U_m are open sets in X and

$$A \subset U_1 \cup \cdots \cup U_m \stackrel{\text{def}}{=} U$$
 and $A \cap U_i \neq \emptyset$ for all $i = 1, \ldots, m$.

By assumption, there exists $n_0 \in \mathbb{N}$ such that $A_n \subset U$ for all $n \geq n_0$. Moreover, since $A \cap U_i \neq \emptyset$, also we get $n_i \in \mathbb{N}$ such that $A_n \cap U_i \neq \emptyset$ for all $n \geq n_i$ for i = 1, ..., m. Therefore $A_n \in O(U_1, ..., U_s)$ for all $n \geq N = \max\{n_i : i = 0, ..., s\}$. This implies that $A_n \to A$.

Finally we will prove (2). Since for every open neighborhood V of any point in A there exists $n_0 \in \mathbb{N}$ such that $A_n \cap V \neq \emptyset$ for all $n \geq n_0$ we get that

$$A \subset \bigcap_{m \ge 1} \overline{\bigcup_{n \ge m} A_n}$$

Reverse content is equivalent to prove that for every compact set K such that $K \cap A = \emptyset$, there exists $n_0 \in \mathbb{N}$ so that $A_n \cap K = \emptyset$ for all $n \ge n_0$. But this is a consequence again of Lemma 2.1. Indeed, since K and K are compact sets, we can find disjoint open sets K and K such that $K \cap K$ are disjoint open sets K and $K \cap K$ by the above characterization of Vietoris convergence, there is $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ by the above characterization of Vietoris convergence, there is $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ are disjoint open sets K and $K \cap K$ such that $K \cap K$ such that K

Now, we will prove Theorems A and B (Items 1 and 2). First we prove Item 1 of Theorem B.

Proposition 2.3. If $x \in \mathcal{B}_p(A)$ then both, A and $\overline{\Gamma(x)}$, are forward invariant compact sets such that

$$A = \bigcap_{m \ge 1} \overline{\bigcup_{n \ge m}} F^n(\{x\})$$
 and $\overline{\Gamma(x)} = \bigcup_{n \ge 1} F^n(\{x\}) \cup A$.

In particular,

 $\overline{\{f_\omega^m(x): m \geq n\}}$ is a compact set for all $n \in \mathbb{N}$ and $\omega \in \Omega$.

Proof. Set $K \stackrel{\text{def}}{=} \overline{\Gamma(x)}$. Since $F^n(\{x\}) \to A$, Lemma 2.2 implies $A \subset K$. Moreover, this lemma also provides the above characterization of A, and consequently of K. From these characterizations, it is easy to conclude that A and K are both forward invariant sets.

Now we will show that K is compact. Let $\{U_{\alpha} : \alpha \in I\}$ be an open cover of K. Since $A \subset K$, by the compactness of A there exists a finite subset J_1 of I such that

$$A\subset\bigcup_{\alpha\in J_1}U_\alpha\stackrel{\mathrm{def}}{=}U.$$

Again, by Lemma 2.2, there exists $n_0 \in \mathbb{N}$ such that the union of $F^n(\{x\})$ for $n \ge n_0$ is contained in U. On the other hand, the set $F(\{x\}) \cup \cdots \cup F^{n_0-1}(\{x\})$ is a finite union of compact sets and thus, it is compact. Hence, there is a finite subset I_2 of I such that

$$F(\lbrace x\rbrace) \cup \cdots \cup F^{n_0-1}(\lbrace x\rbrace) \subset \bigcup_{\alpha \in J_2} U_{\alpha}.$$

Put together all and setting $J = J_1 \cup J_2$ we get that

$$K = \overline{\Gamma(x)} = A \cup F(\{x\}) \cup \cdots \cup F^{n_0 - 1}(\{x\}) \cup \bigcup_{n \ge n_0} F^n(\{x\}) \subset \bigcup_{\alpha \in J} U_\alpha$$

concluding that *K* is a compact.

Remark 2.4. With the same proof one shows that if A is a strict attractor and $S \subset \mathcal{B}(A)$ is compact then the closure of $\Gamma(S)$ is also a compact set.

Now, we characterize the forward minimal self-similar compact sets (Item 1 of Theorem A).

Proposition 2.5. A compact set A is a forward minimal self-similar set if and only if $A \subset \mathcal{B}_{\nu}(A)$.

Proof. Assume that A is forward minimal self-similar compact set. Hence for every $x \in A$, the closure of $\Gamma(x)$ is equal to A. On the other hand, for any basic neighborhood $O\langle U_1, \ldots, U_s \rangle$ of A in $\mathscr{H}(X)$ we have that $F^n(\{x\}) \subset A \subset U_1 \cup \cdots \cup U_s$ for all $n \geq 1$ and, by the density of $\Gamma(x)$, $F^n(\{x\}) \cap U_i \neq \emptyset$ for any n large enough and $i = 1, \ldots, s$. Then $F^n(\{x\}) \in O\langle U_1, \ldots, U_s \rangle$ for any n sufficiently large and thus $F^n(\{x\}) \to A$. Therefore $x \in \mathcal{B}_p(A)$.

Assume now that $A \subset \mathcal{B}_p(A)$ and consider $K = \overline{\Gamma(x)}$ for some $x \in A$. By Proposition 2.3, K is a compact Hausdorff topological space so that $A \subset K$ and $F(K) \subset K$. Thus we can restrict the map F to the set of non-empty compact set of K. According to [14, Prop. 1.5.3 (iv)], the Hutchinson operator $F : \mathcal{K}(K) \to \mathcal{K}(K)$ is continuous and since $F^n(\{x\}) \to A$, we get that

$$A = \lim_{n \to \infty} F^n(\{x\}) = F(\lim_{n \to \infty} F^{n-1}(\{x\})) = F(A).$$

Now, we will prove that A is forward minimal. By Proposition 2.3, $A \subset \overline{\Gamma(x)}$ for all $x \in A \subset \mathcal{B}_p(A)$. On the other hand, since F(A) = A then $\Gamma(x) \subset A$ for all $x \in A$. Both inclusions imply the density on A of the orbit of any point $x \in A$ and then the forward minimality of A.

We complete the proof of Theorem A by studying the basin of attraction of a strict attractor.

Proposition 2.6. *If A is a strict attractor, then*

- (1) $F^n(K) \to A$ in the Vietoris topology for all compact set $K \subset \mathcal{B}(A)$;
- (2) A is a pointwise attractor and $\mathcal{B}(A) = \mathcal{B}_p(A)$.

Proof. The first item is consequence of Lemma 2.1. Indeed, given any compact set K in $\mathcal{B}(A)$, by compactness we can find open neighborhoods U_1, \ldots, U_s of A such that $K \subset U_1 \cup \cdots \cup U_s$ and $F^n(S) \to A$ for any compact set S in U_i for all $i = 1, \ldots, s$. By Lemma 2.1 there are compact sets $K_i \subset U_i$ for $i = 1, \ldots, m$ such that $K = K_1 \cup \cdots \cup K_s$. Then, $F^n(K) = F^n(K_1) \cup \cdots \cup F^n(K_s)$ and thus $F^n(K)$ converges to A.

Now we will prove the second item. By means of the first item, $\mathcal{B}(A) \subset \mathcal{B}_p(A)$. Thus, since $\mathcal{B}(A)$ is an open set containing A we get that A is a pointwise attractor. To conclude, we will show that $\mathcal{B}_p(A) \subset \mathcal{B}(A)$. Given $x \in \mathcal{B}_p(A)$ we want to prove that x belongs to $\mathcal{B}(A)$.

Claim 2.7. If there exists a neighborhood V of x such that $F^n(K) \to A$ in the Vietoris topology for all compact set $K \subset V$ then $x \in \mathcal{B}(A)$.

Proof. Since *A* is an attractor there exists a neighborhood U_0 of *A* such that $F^n(S) \to A$ for all compact set *S* in U_0 . Take $U = U_0 \cup V$. Clearly, *U* is a neighborhood of *A* and $x \in U$. On the other hand, by Lemma 2.1, any compact set *K* in *U* can be written as the union of two compact set K_0 and K_1 contained in U_0 and *V* respectively. Now, since $F^n(K) = F^n(K_0) \cup F^n(K_1)$ it follows that $F^n(K)$ converges to *A* for all compact set *K* in the neighborhood *U* of *A*. This implies that $x \in \mathcal{B}(A)$. □

Now, we will get a neighborhood V of x in the assumptions of the above claim. Since $\mathcal{B}(A)$ is an open neighborhood of A and $x \in \mathcal{B}(A)$, we get $m \in \mathbb{N}$ such that $F^m(\{x\}) \subset \mathcal{B}(A)$. Equivalently, $f_{\omega_m} \circ \cdots \circ f_{\omega_1}(x) \in \mathcal{B}(A)$ for all $\omega_i \in \{1, \dots, k\}$ for $i = 1, \dots, m$. By the continuity of the generator f_1, \dots, f_k of the IFS, we get an open set V such that $x \in V$ and $f_{\omega_m} \circ \cdots \circ f_{\omega_1}(V) \subset \mathcal{B}(A)$ for all $\omega_i \in \{1, \dots, k\}$ for $i = 1, \dots, m$. In particular, for every compact set K in V it holds that $F^m(K) \subset \mathcal{B}(A)$ and thus, by the first item, $F^n(K)$ converges to A.

To end this section we will prove Item 2 of Theorem B.

Proposition 2.8. Consider $\omega \in \Omega$ and $x \in \mathcal{B}_p(A)$. Then $A \subset \overline{O^+_{\omega}(x)}$ if and only if

$$\lim_{n\to\infty} \overline{\{f_{\omega}^m(x): m\geq n\}} = A \text{ in the Vietoris topology}$$

and if and only if

$$A = \bigcap_{n \ge 1} \overline{\{f_{\omega}^m(x) : m \ge n\}}.$$

Proof. According to Proposition 2.3, $A_n = \overline{\{f_\omega^m(x) : m \ge n\}}$ is a compact set for all $n \in \mathbb{N}$. Moreover, $A_{n+1} \subset A_n$ and hence, by Lemma 2.2, if $A_n \to A$ in the Vietoris topology,

$$A = \bigcap_{n>1} A_n \subset A_1 = \overline{O_{\omega}^+(x)}.$$

Reciprocally, let U and V be open set such that $A \subset U$ and $V \cap A \neq \emptyset$. Since $x \in \mathcal{B}_p(A)$ then $F^n(\{x\}) \to A$ and thus by Lemma 2.2 there exists $n_0 \in \mathbb{N}$ such that

$$\bigcup_{n\geq n_0} F^n(\{x\}) \subset U.$$

In particular, the union of A_n for $n \ge n_0$ is contained in U. Moreover, since $A \subset \overline{O_{\omega}^+(x)}$ we have $A_n \cap V \ne \emptyset$ for all n large enough. Lemma 2.2 implies that $A_n \to A$ in the Vietoris topology completing the proof.

3. Deterministic chaos game

3.1. **Equivalence.** We will conclude the proof of Theorem B.

Proposition 3.1. Let A be a pointwise attractor. Then the following are equivalent:

- (1) there exists $\omega \in \Omega$ such that $A \subset \overline{O_{\omega}^+(x)}$ for all $x \in \mathcal{B}_p(A)$;
- (2) $A \subset \overline{O_{\omega}^{+}(x)}$ for all disjunctive sequence $\omega \in \Omega$ and $x \in \mathcal{B}_{p}(A)$;
- (3) there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$A \subset \overline{O_{\omega}^+(x)}$$
 for all $\omega \in \Omega_0$ and $x \in \mathcal{B}_p(A)$.

Proof. It suffices to show that (1) implies (2). Let x be a point in $\mathcal{B}_p(A)$. According to Proposition 2.3, $K = \overline{\Gamma(x)}$ is a forward invariant compact set with $A \subset K$. Moreover, since A is a pointwise attractor we also obtain that $K \subset \mathcal{B}_p(A)$. The following claim will be useful to prove the density of disjunctive fiberwise orbits:

Claim 3.2. Let Z be a forward invariant set such that $A \subset Z$. If for any non-empty open set $I \subset X$ with $A \cap I \neq \emptyset$, there exists $f_{i_n} \circ \cdots \circ f_{i_1} \in \Gamma$ such that

for each
$$z \in Z$$
 there is $t \in \{1, ..., n\}$ so that $f_{i_t} \circ \cdots \circ f_{i_1}(z) \in I$

then

$$A \subset \overline{O_{\omega}^+(x)}$$
 for all disjunctive sequence $\omega \in \Omega$ and $x \in Z$.

Proof. Consider any open set I such that $A \cap I \neq \emptyset$, $x \in Z$ and a disjunctive sequence $\omega \in \Omega$. Using the fact that ω is a disjunctive sequence and that Z is a forward invariant set we can choose $m \geq 1$ such that

$$[\sigma^m(\omega)]_j = i_j$$
 for $j = 1, ..., n$ and $z = f_\omega^m(x) \in Z$.

Hence, by assumption, there exists t = t(z) such one has that $f_{\omega}^{m+t(z)}(x) \in I$ which proves the density on A of the ω -fiberwise orbit of x.

Notice that $F(K) \subset K$ and hence we can take as Z = K in the above claim. Let I be an open set so that $I \cap A \neq \emptyset$. By assumption, since $Z \subset \mathcal{B}_p(A)$, there exists a sequence $\omega \in \Omega$ such that for each point $z \in Z$ the ω -fiberwise orbit of z is dense in A. In particular, there is $n = n(z) \in \mathbb{N}$ such that $\{f_\omega^m(z) : m \leq n\} \cap I \neq \emptyset$. By continuity of the generators f_1, \ldots, f_k of the IFS, there exists an open neighborhood V_z of z such that $\{f_\omega^m(y) : m \leq n\} \cap I \neq \emptyset$ for all $y \in V_z$. Then, the compactness of Z implies that we can extract open sets V_1, \ldots, V_m and positive integer n_1, \ldots, n_m such that $Z \subset V_1 \cup \cdots \cup V_m$ and $\{f_\omega^m(z) : m \leq n_i\} \cap I \neq \emptyset$ for all $z \in V_i$ and $i = 1, \ldots, m$. Hence the assumptions of Claim 3.2 hold taking $f_{\omega_n} \circ \cdots \circ f_{\omega_1} \in \Gamma$ where $n = \max\{n_i : i = 1, \ldots, m\}$. Therefore, since the initial point $x \in \mathcal{B}_p(A)$ belongs to Z, we conclude that any disjunctive fiberwise orbit of x is dense in A and complete the proof. \square

3.2. **Necessary condition.** We will prove Theorem D.

Proof of Theorem D. Clearly if there exists a minimal orbital branch, i.e., $\omega = \omega_1 \omega_2 \cdots \in \Omega$ such that $O_{\omega}^+(x)$ is dense for all x, then the IFS is forward minimal. We will assume that it is not backward minimal. Then, there exists a non-empty closed set $K \subset X$ such that $\emptyset \neq f^{-1}(K) \subset K \neq X$ for all $f \in \Gamma$. We can consider

$$K_n = \bigcap_{i=1}^n f_{\omega_1}^{-1} \circ \cdots \circ f_{\omega_i}^{-1}(K) = f_{\omega_1}^{-1} \circ \cdots \circ f_{\omega_n}^{-1}(K) \quad \text{and} \quad K_\omega = \bigcap_{n=1}^\infty K_n.$$

Hence K_n is a nested sequence of closed sets. By assumption of this theorem, the space X where the IFS is defined is a compact Hausdorff topological space. As a consequence, K_{ω} is not empty and then for every $x \in K_{\omega}$ we have that $O_{\omega}^+(x) \subset K$. Since K is not equal to X it follows that there exists a point $x \in X$ so that the ω -fiberwise orbit of x is not dense. But this is a contradiction and we conclude the proof.

As in the introduction we notified, an IFS is forward minimal if and only if every point has dense Γ -orbit. To complete the section we want to point out the following straightforward similar equivalent definition of backward minimality.

Lemma 3.3. Consider an IFS of surjective continuous maps of a topological space X. Then the IFS is backward minimal if and only if

$$X = \overline{\Gamma^{-1}(x)}$$
 for all $x \in X$

where

$$\Gamma^{-1}(x) \stackrel{\text{def}}{=} \{ y \in X : \text{there exists } g \in \Gamma \text{ such that } g(y) = x \}.$$

3.3. **Deterministic chaos game on the circle.** In [10, Thm. A] it was proved that every forward and backward minimal IFS of preserving-orientation homeomorphisms of the circle satisfies the deterministic chaos game. However, the assumption of preserving-orientation can be removed from this statement as we explain below. The main tool in the proof of the above result was Antonov's Theorem [1] (see [10, Thm. 2.1]). This theorem is statement for preserving-orientation homeomorphisms of the circle. Supported in this result the authors showed a key lemma (see [10, Lem. 2.2]) to prove the above statement. In fact, in this lemma, through Antonov's result, is the unique point in the proof where the preserving-orientation assumption is used. This lemma can be improved removing the preserving orientation assumption by two different ways. The first is observing that in fact, this assumption is not necessarily in the original proof of Antonov as easily one can follow from the argument described in [13, proof of Theorem 2]. Another way is to use the recently generalization of Antonov's result [16, Thm. D] instead the key lemma above mentioned. In any case, we get that every forward and backward minimal IFS of homeomorphisms of the circle satisfies the deterministic chaos game. That is, (3) implies (1) in Corollary E. On the other hand, (1) implies (3) follows from Theorem C. Finally, to complete the proof of Corollary E it suffices to note that according to Theorem B, (1) and (2) are equivalent.

We will prove now Corollary F. As in the introduction we mentioned, for ordinary dynamical systems, the minimality of a map T is equivalent to that of T^{-1} . Nevertheless this is not the case for dynamical systems with several maps as Kleptsyn and Nalskii pointed at [15, pg. 271]. However, they omitted to include these examples of forward but not backward minimal IFSs. Hence, to provide a complete proof of Corollary F we will show that indeed such IFSs of homeomorphisms of S^1 can be constructed.

3.3.1. Forward but not backward minimal IFSs on the circle. Consider a group of homeomorphisms of the circle. Then, there can occur only one of the following three options [19]: existence of a finite orbit, every orbit is dense on the circle, or there exists a unique minimal Cantor set invariant by the group action. Minimal here means that all the points in the invariant set have dense orbits. If there exists a minimal invariant Cantor set, this set is called *exceptional minimal set*. The following proposition is stated in [19, Exercice 2.1.5]:

Proposition 3.4. There exists a finitely generated group G of diffeomorphisms of S^1 admitting an exceptional minimal set K such that the orbit by G of every point of $S^1 \setminus K$ is dense in S^1 .

Recall that a set *A* is *invariant by G* if g(A) = A for all $g \in G$. We will use the following:

Lemma 3.5. Let G and K be as in Proposition 3.4. Then the unique closed subsets of S^1 invariant by G are \emptyset , K and S^1 .

Proof. Let B be a closed subset of S^1 invariant by G. If $B \neq \emptyset$, then $K \subset B$ by minimality of K, and if $B \neq K$, it means that B contains a point x in $S^1 \setminus K$, and by invariance, B contains the orbit of x by G which is dense, hence $B = S^1$.

We say that S is a *symmetric generating system* of a group G if G is generated by S as a semigroup. On the other hand, any two Cantor set are homemorphic. In fact, if K_I and K_J are two Cantor sets in an interval I and J respectively, there exists a homeomorphisms $g: I \to J$ so that $g(K_I) = K_J$ (see for instance [2]). Hence given any Cantor set K in S^1 one can find a homeomorphisms K0 of K1 so that K2 is strictly contained in K3 (or K3) strictly contains K3.

Theorem 3.6. Let G and K be as in Proposition 3.4 and f_1, \ldots, f_n be a symmetric system of generators of G. Consider any homeomorphisms h of S^1 such that h(K) strictly contains K. Then the IFS generated by f_1, \ldots, f_n , h is forward minimal but not backward minimal.

Proof. Let $K_1 \stackrel{\text{def}}{=} h(K)$. By assumption $K \subsetneq K_1$. We claim that the IFS generated by f_1, \ldots, f_n, h is forward minimal but not backward minimal.

- The IFS is not backward minimal: since K is invariant by the group G, $f_i^{-1}(K) = K$ for i = 1, ..., n. We also have $h^{-1}(K) \subset h^{-1}(K_1) = K$. Thus, K is forward invariant by $f_1^{-1}, ..., f_n^{-1}, h^{-1}$ and so the IFS is not backward minimal.
- The IFS is forward minimal: let $B \subset S^1$ be a forward invariant by f_1, \ldots, f_n, h closed set. In particular B is invariant by G, hence $B \in \{\emptyset, K, S^1\}$ by Lemma 3.5. Moreover $B \neq K$ since K is not invariant by h (otherwise $K_1 = h(K) = h(B) \subset B = K$ but K_1 strictly contains K). So, $B \in \{\emptyset, S^1\}$, which means that the IFS is forward minimal.

3.4. **Deterministic chaos game for contractible attractors.** In what follows, *A* denotes a pointwise attractor. We start studying the relation between strongly-fibred and contractible.

Proposition 3.7. If A is strongly-fibred then it is contractible. Moreover, if in addition A is a strict attractor then for every compact set K in $\mathcal{B}(A)$ and every open set U so that $A \cap U \neq \emptyset$ there exists $g \subset \Gamma$ such that $g(K) \subset U$.

Proof. Consider a compact set K in A and let U be any open set such that $A \cap U \neq \emptyset$. Since A is strongly-fibred, we get $\omega \in \Omega$ such that

$$A_{\omega} = \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A) \subset U.$$

Notice that since $f_i(A) \subset A$ for i = 1, ..., k then $f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A)$ is a nested sequence of compact sets and thus, for n large enough, $f_{\omega_1} \circ \cdots \circ f_{\omega_n}(A) \subset U$. In particular, taking $h = f_{\omega_1} \circ \cdots \circ f_{\omega_n} \in \Gamma$ we have that $h(K) \subset U$. This proves that A is contractible.

We will assume now that A is a strict attractor and consider K in $\mathcal{B}(A)$. As above we have that $h(A) \subset U$. We claim that there exists a neighborhood V of A such that $h(V) \subset U$. Indeed, it suffices to note that h is a continuous map and hence $h^{-1}(U)$ is an open set containing the compact set A. Since A is a strict attractor, $F^n(K) \to A$ in the Vietoris topology and in particular, there is $f \in \Gamma$ such that $f(K) \subset V$. Thus, taking $g = h \circ f \in \Gamma$, it follows that $g(K) \subset h(V) \subset U$.

Remark 3.8. If A is strongly fibred we have proved that one can contract any compact set in A. In particular we can contracts A and this implies that there exists some generator f_i such that $f_i(A) \neq A$.

Now, we give an example of an IFS defined on S^1 whose unique attractor is the whole space (that is the IFS is minimal) and it is contractible but not strongly-fibred. This example shows that these two properties are not equivalent.

Example 3.9. Consider the IFS generated by two diffeomorphisms g_1, g_2 , where g_1 is rotation with irrational rotation number and g_2 is an orientation preserving diffeomorphism with a unique fixed point p such that $Dg_2(p) = 1$ and α -limit set and ω -limit set of each point $q \in S^1$ is equal to $\{p\}$. Clearly, the IFS acts minimally on S^1 and thus $A = S^1$ is the attractor. Since g_1 and g_2 map S^1 onto itself, it follows that for each $\omega \in \Omega$, the fiber $A_\omega = S^1$. This implies that S^1 is not strongly-fibred, but it still contractible. Indeed, let K any compact so that $K \neq S^1$. Then, there is an open arc J of S^1 which is not dense in S^1 such that $K \subset J$. If J contains the fixed point p, there is an integer n such that $g_1^n(J)$ does not contain p. So, without less of generality, we may assume that $p \notin J$. Now, it is easy to see that $g_2^k(J)$ tends to p as $k \to \infty$. This implies that S^1 is contractile.

Above example is based in the fact that the attractor A satisfies that $f_i(A) = A$ for all i = 1, ..., k. The above proposition and the following show that if $f_i(A)$ is not equals to A for some generator f_i then both properties are equivalent.

Proposition 3.10. *If* A *is contractible and* $f_i(A) \neq A$ *for some* $1 \leq i \leq k$ *then* A *is strongly-fibred.*

Proof. First of all note that it suffices to prove that for any open set U with $U \cap A \neq \emptyset$, there is $h \in \Gamma$ so that $h(A) \subset U$. To this end, notice that since A is an attractor then the action of Γ restricted to A is minimal. Then, there exist $h_1, \ldots, h_m \in \Gamma$ so that $A \subset h_1^{-1}(U) \cup \cdots \cup h_m^{-1}(U)$. On the other hand, by assumption, there is $i \in \{1, \ldots, k\}$ such that $f_i(A) \neq A$. Hence $f_i(A)$ is a compact set strictly contained in A and since A is contractible there exist $g \in \Gamma$ and $j \in \{1, \ldots, m\}$ such that $g(f_i(A)) \subset h_j^{-1}(U)$. Thus, taking $h = h_j \circ g \circ f_i \in \Gamma$, it follows that $h(A) \subset U$ concluding the proof.

In order to proof Theorem G, we first need a lemma (compare with Claim 3.2). Here we understand $f_{i_t} \circ \cdots \circ f_{i_1}$ for t = 0 as the identity map.

Lemma 3.11. If for any non-empty open set $I \subset X$ with $A \cap I \neq \emptyset$, there exist a neighborhood Z of A and $f_{i_s} \circ \cdots \circ f_{i_1} \in \Gamma$ such that

for each
$$z \in Z$$
 there is $t \in \{0, ..., s\}$ so that $f_{i_t} \circ \cdots \circ f_{i_1}(z) \in I$

then

$$A \subset \overline{O_{\omega}^+(x)}$$
 for all disjunctive sequence $\omega \in \Omega$ and $x \in \mathcal{B}_p(A)$.

Proof. Consider any open set I such that $A \cap I \neq \emptyset$ and $x \in \mathcal{B}_p(A)$ and disjunctive sequence $\omega \in \Omega$. Using the fact that ω is a disjunctive sequence and that Z is a neighborhood of A and $F^n(\{x\}) \to A$ if $x \in \mathcal{B}_p(A)$, we can choose $m \ge 1$ such that

$$[\sigma^m(\omega)]_j = i_j \text{ for } j = 1, \dots, s \text{ and } z = f_\omega^m(x) \in Z.$$

Hence, by assumption, there exists t = t(z) such one has that $f_{\omega}^{m+t(z)}(x) \in I$ which proves the density on A of the ω -fiberwise orbit of x.

The following result proves the first part in Theorem G.

Proposition 3.12. *If A is a contractible then it satisfies the deterministic chaos game.*

Proof. In order to apply Lemma 3.11, we consider any non-empty open set I with $I_A \stackrel{\text{def}}{=} A \cap I \neq \emptyset$. Hence, $K = A \setminus I_A$ is a compact set so that $K \neq A$. Since A is an attractor, the action of Γ restrict to A is minimal and thus, there exist $h_1, \ldots, h_m \in \Gamma$ such that $A \subset h_1^{-1}(I) \cup \cdots \cup h_m^{-1}(I)$. On the other hand, since A is contractible, there exist $i \in \{1, \ldots, m\}$ and $g \in \Gamma$ such that $g(K) \subset h_i^{-1}(I)$. By continuity of the generators, and thus, of $h = h_i \circ g$, we find an open set U with $K \subset U$ such that $h(U) \subset I$. Take, $Z = U \cup I$ and $f_{i_s} \circ \cdots \circ f_{i_1} = h \in \Gamma$. Clearly, Z is open with $A = K \cup I_A \subset U \cup I = Z$ and for every $z \in Z$, there is $t \in \{0, s\}$ such that $f_{i_t} \circ \cdots \circ f_{i_1}(z) \in I$. Lemma 3.11 implies A satisfies the deterministic chaos game.

Now, we conclude the proof of Theorem G. To do this, first we recall that the skew-product Φ associated with the IFS generated by $\mathscr{F} = \{f_1, \dots, f_k\}$ is given by

$$\Phi(\omega, x) = (\sigma(\omega), f_{\omega_1}(x)), \quad \text{where } \omega = \omega_1 \omega_2 \cdots \in \Omega = \{1, \dots, k\}^{\mathbb{N}} \text{ and } x \in X.$$

Here *σ* denotes the lateral shift on Ω . If the generators f_1, \ldots, f_k are homeomorphisms we can consider the two-side skew-product $\tilde{\Phi}: \Sigma \times X \to \Sigma \times X$ defined as follows:

$$\tilde{\Phi}(\omega, x) = (\sigma(\omega), f_{\omega_0}(x)), \quad \text{where } \omega = \dots \omega_{-1}; \omega_0 \omega_1 \dots \in \Sigma = \{1, \dots, k\}^{\mathbb{Z}} \text{ and } x \in X.$$

Here σ denotes the lateral shift on Σ . This skew-product $\tilde{\Phi}$ is an homeomorphism where

$$\tilde{\Phi}^{-1}(\omega, x) = (\sigma^{-1}(\omega), f_{\omega_{-1}}^{-1}(x)), \quad \text{for all } (\omega, x) \in \Sigma \times X.$$

In order to prove the second part of Theorem G, we can restrict ours attention to $\Phi|_{\Omega\times A}$, or in other words, we can assume that we are working with a forward minimal strong-fibred or invertible IFS on a compact Hausdorff topological space. Recall that an IFS is said to be *invertible* if its generators are homeomorphisms (cf. [8, Definition 2.5]). A forward minimal IFS is said to be strongly-fibred (resp. contractible) is the whole space is strongly-fibred (resp. contractible) attractor. Finally, we denote by Σ^+_{disj} the set of bi-lateral sequences in Σ having dense forward orbit under the shift map. With this notations and observations we will conclude Theorem G as consequence of the following result:

Proposition 3.13. Consider a contractible forward minimal IFS generated by continuous maps of a compact Hausdorff topological space A. Assume the IFS is either, strongly-fibred or invertible. Then

$$\Omega \times A = \overline{\{\Phi^n(\omega, x) : n \in \mathbb{N}\}}$$
 for all disjunctive sequence ω and $x \in A$.

Moreover, in the case that the IFS is invertible, every point in $\Sigma_{disi}^+ \times A$ is recurrent for $\tilde{\Phi}$.

Proof. Let $\omega \in \Omega$ be a disjunctive sequence and consider $x \in A$. We want to show that (ω, x) has dense orbit in $\Omega \times A$ under Φ . In order to prove this, let $C_{\alpha}^+ \times I$ be a basic open set of $\Omega \times A$. That is, C_{α}^+ is a cylinder in Ω around of a finite word $\alpha = \alpha_1 \dots \alpha_\ell$ and I is an open set in A which we can assume that it is not equal to the whole space. It suffices to prove that there exists an iterated by Φ of (ω, x) that belongs in $C_{\alpha}^+ \times I$. To do this, similarly as in the previous proposition, we use the forward minimality of Γ on A to find maps $h_1, \dots, h_m \in \Gamma$ such that $A = h_1^{-1}(I) \cup \dots \cup h_m^{-1}(I)$. Set $f = f_{\alpha_\ell} \circ \dots \circ f_{\alpha_1} \in \Gamma$.

Assume first that the IFS is strongly-fibred. Then there exists a generator f_i such that $K = f_i(A) \neq A$. By Proposition 3.7, the IFS is contractible and thus we find $g \in \Gamma$ such that $g(K) \subset h_\ell^{-1}(I)$ for some $\ell \in \{1, \ldots, m\}$. Hence, $h(K) \subset I$ where $h = h_\ell \circ g$. Let $f \circ h \circ f_i = f_{i_s} \circ \cdots \circ f_{i_1}$. Since $\omega \in \Omega$ is a disjunctive sequence, we can choose $m \geq 1$ such that

$$[\sigma^m(\omega)]_j = i_j \text{ for } j = 1, \dots, s.$$
 (3)

Set $z = f_{\omega}^m(x)$. Then, $h \circ f_i(z) \in I$. Moreover,

$$\Phi^{m+t}(\omega,x)=\Phi^t(\sigma^m(\omega),z)=(\sigma^{m+t}(\omega),h\circ f_i(z))\in C_\alpha^+\times I$$

where t = 1 + |h| being |h| the length of h with respect to the set of generators f_1, \ldots, f_k .

Now, assume that the contractible forward minimal IFS is also invertible. Hence f is an homeomorphisms of A and thus $\emptyset \neq f(I) \neq A$ is an open set. Let $K = A \setminus f(I)$. Notice that K is a non-empty compact set different of A and by means of the contractibility of the IFS we get $g \in \Gamma$ so that $h(K) \subset I$ where $h = h_{\ell} \circ g$ for some $\ell \in \{1, \ldots, m\}$. Let $f \circ h \circ f = f_{i_s} \circ \cdots \circ f_{i_1}$.

Similar as above, since $\omega \in \Sigma_{disj}^+$ is a forward disjunctive bi-lateral sequence we choose $m \ge 1$ satisfying (3) and denote $z = f_\omega^m(x)$. If $z \in I$, $\Phi^m(\omega, x) = (\sigma^m(\omega), z) \in C_\alpha^+ \times I$. Otherwise, $f(z) \in K$ and then $h \circ f(z) \in I$ and thus

$$\Phi^{m+t}(\omega, x) = \Phi^t(\sigma^m(\omega), z) = (\sigma^{m+t}(\omega), h \circ f(z)) \in C^+_{\alpha} \times I$$

where t = |f| + |h| being |f| and |h| the length of f and h respectively.

Finally, to prove the last claim we need to see that if U is neighborhood of $(\omega, x) \in \Sigma_{disj}^+ \times A$ then there is a forward iteration of U by $\tilde{\Phi}$ meets U. Equivalently, one can so that a forward iteration of U meets backward iteration of the same neighborhood. Observe that the backward iteration of U contains a set of the form $(\Sigma^- \oplus C_\alpha^+) \times I$ where I is an open interval of A and $\Sigma^- \oplus C_\alpha^+$ is the set of sequences $(\omega^-; \omega^+) \in \Sigma$ with $\omega^- = (\omega_{-i})_{i \in \mathbb{N}}$ arbitrary and $\omega^+ = (\omega_i)_{i \in \mathbb{Z}_+}$ belongs in the (unilateral) cylinder C_α^+ . Thus, to get the recurrence of (ω, x) for the map $\tilde{\Phi}$ is enough to prove that the point (ω^+, x) has dense orbit under Φ . This is followed from the fact that $\omega \in \Sigma_{disj}^+$ and the previous item and we conclude the proof of the theorem.

We end the section proving Corollary H. Namely, we will show the following:

Corollary 3.14. Consider a forward and backward minimal IFS of homeomorphisms of a metric space X and assume that there is a map h in the semigroup Γ generated by these maps with exactly two fixed points, one attracting and one repelling. Then X is a contractible strict attractor and consequently satisfies the deterministic chaos game.

Proof. The forward minimality implies that X is a strict attractor. Consider now any compact set $K \subset X$ such that $K \neq X$. By the backward minimality there exist $T_1, \ldots, T_s \in \Gamma$ such that

$$X = \bigcup_{i=1}^{s} T_i(X \setminus K).$$

Let p and q be, respectively, the attracting and the repelling fixed points of h. Then there is $i \in \{1, ..., s\}$ so that $q \in T_i(X \setminus K)$. Therefore, $q \notin T_i(K)$ and then the diameter of $h^n \circ T_i(U)$ converges to zero. This shows that the action is contractible.

Acknowledgments

The authors would like to thank Carlos Meniño for the useful conversations during the preparation of this article. The first author was partially supported by the fellowships MTM2014-56953-P project (Spain).

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